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Solution for 'Topics in complex analysis'

(17/09/2025)

H 2.1 (An example of propagation of convergence)
Let $D \subset \mathbb{C}$ be a bounded domain and denote its closure by $\overline{D}$. Let $f_n : \overline{D} \to \mathbb{C}$ be a sequence of continuous functions such that each f_n is holomorphic in D. Assume that the sequence f_n converges uniformly on the boundary ∂D. Show that the sequence f_n converges uniformly on the whole set $\overline{D}$ to some $f : \overline{D} \to \mathbb{C}$.
Hint: Show that f_n is a Cauchy-sequence with respect to uniform convergence.

Solution H 2.1: Let $m \geq n$. By the maximum principle each function $|f_m - f_n|$ attains its maximum on ∂D, so that

$$\sup_{z \in \overline{D}} |f_m(z) - f_n(z)| = \sup_{z \in \partial D} |f_m(z) - f_n(z)|.$$

By assumption, the sequence $f_n|_{\partial D}$ is a Cauchy sequence with respect to uniform convergence. Thus the above equality shows that f_n is a Cauchy sequence with respect to uniform convergence on $\overline{D}$. In particular, it converges uniformly to a continuous function $f : \overline{D} \to \mathbb{C}$ (which is also holomorphic on D). $\square$

H 2.2 (On sequences of holomorphic functions)
a) Let $f_n : B_1(0) \to \mathbb{C}$ be a sequence of holomorphic functions that converges locally uniformly to a holomorphic function $f : B_1(0) \to \mathbb{C}$. Does the sequence $f_n^{(n)}$ converge locally uniformly to some continuous function $g : B_1(0) \to \mathbb{C}$? Give a proof or find a counterexample.

b) Give an example of an open set $U \subset \mathbb{C}$ and a sequence of holomorphic functions $f_n : U \to \mathbb{C}$ that converges locally uniformly to some $f : U \to \mathbb{C}$ and such that each f_n has exactly one zero, while f has no zeros.

Solution H 2.2: a) The claim is false. Consider the sequence $f_n(z) = z^n$, which converges locally uniformly to 0 on $B_1(0)$. Then the n-th derivative is given by $f_n^{(n)}(z) = n!$, which is not even bounded as $n \to +\infty$.

b) We take $U = \mathbb{C}$ and $f_n(z) = \frac{z}{n} + 1$. Then each f_n has exactly one zero (at $-n$), but the sequence converges locally uniformly to $f(z) = 1$, which has no zeros. For an example with uniform convergence (rather than just locally uniform), one can for instance take $f_n(z) = z^2 - \frac{1}{n}$ on $U = \{\mathrm{Re}\,(z) > 0\}$. $\square$

H 2.3 (Convergence of varying path-integrals)
Let $f_n : U \to \mathbb{C}$ be a sequence of holomorphic functions that converges locally uniformly on an open set $U \subset \mathbb{C}$ to some $f : U \to \mathbb{C}$. Let further $\gamma_n : [0,1] \to U$ be a sequence of C^1-paths such that $\gamma_n \to \gamma$ and $\gamma_n' \to \gamma'$ uniformly on $[0,1]$ for some C^1-path $\gamma : [0,1] \to U$. Show that

$$\lim_{n \to +\infty} \int_{\gamma_n} f_n(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z.$$

Solution H 2.3: The definition of the complex path integral is

$$\int_{\gamma_n} f_n(z)\,\mathrm{d}z = \int_0^1 f_n(\gamma_n(t))\gamma_n'(t)\,\mathrm{d}t.$$

By the dominated convergence theorem, it suffices to show that the integrand in the right converges pointwise to $f(\gamma(t))\gamma'(t)$, and that this function is integrable in $[0,1]$. We start with the first task.
By assumption, the functions γ_n' converge (in particular) pointwise to γ'. Next, note that for any $t \in [0,1]$ we have

$$|f_n(\gamma_n(t)) - f(\gamma(t))| \le |f_n(\gamma_n(t)) - f(\gamma_n(t))| + |f(\gamma_n(t)) - f(\gamma(t))|.$$

Since $\gamma_n(t) \to \gamma(t)$ in U, the continuity of f implies that $f(\gamma_n(t)) \to f(\gamma(t))$. Thus we just need to control the first difference on the right hand side, where the local uniform convergence will be used. To this end, we have to ensure that the sequence $\gamma_n(t)$ stays in a compact subset of U. Since $\gamma_n(t) \to \gamma(t) \in U$, it suffices to choose $r > 0$ such that $\overline{B_r(\gamma(t))} \subset U$. Then for n sufficiently large we have $\gamma_n(t) \in B_r(\gamma(t))$, hence by the locally uniform convergence of the f_n we have

$$|f_n(\gamma_n(t)) - f(\gamma_n(t))| \le \sup_{z \in \overline{B_r(\gamma(t))}} |f_n(z) - f(z)| \to 0 \quad \text{as } n \to +\infty.$$

Thus we have shown that $f_n(\gamma_n(t))\gamma_n'(t) \to f(\gamma(t))\gamma'(t)$ pointwise in t, as $n \to +\infty$. In order to pass to the limit in the integral we apply Lebesgue's dominated convergence theorem. Since $\gamma_n' \to \gamma'$ uniformly on $[0,1]$, and γ' is continuous, there exists a constant $C_1 < +\infty$ such that

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0,1]} |\gamma_n'(t)| \le C_1.$$

In order to control the term $f_n(\gamma_n(t))$ we have to ensure that all the paths $\gamma_n([0,1])$ stay in one fixed compact subset of U, which will finish the proof.
First note that γ_n is uniformly bounded (via the same argument used for γ_n' above), so that there exists a constant $\bar{\gamma} < +\infty$ such that

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0,1]} |\gamma_n(t)| \le \bar{\gamma}.$$

Assume by contradiction that there exists a sequence $t_n \in [0,1]$ and a point $\bar{z} \in \partial U$ such that (after passing to a subsequence) we have $\gamma_n(t_n) \to \bar{z}$. Since $\gamma : [0,1] \to U$ has compact image, we know that there exists $\delta > 0$ such that $|\gamma(t) - \bar{z}| \ge \delta$ for all $t \in [0,1]$. But then

$$\delta \le |\gamma(t_n) - \bar{z}| \le |\gamma(t_n) - \gamma_n(t_n)| + |\gamma_n(t_n) - \bar{z}| \le \sup_{t \in [0,1]} |\gamma(t) - \gamma_n(t)| + |\gamma_n(t_n) - \bar{z}|.$$

Note that the right hand side terms converge to 0 as $n \to +\infty$, which yields a contradiction. Hence there exists $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that

$$\bigcup_{n \ge n_0} \gamma_n([0,1]) \subset \{z \in U : |z| \le \bar{\gamma},\, \mathrm{dist}(z, \partial U) \ge \varepsilon\} =: K.$$

The set K is a compact subset of U, hence there exists a constant $C_2 < +\infty$ such that

$$\sup_{n \ge n_0} \sup_{t \in [0,1]} |f_n(\gamma_n(t))| \le \sup_{n \ge n_0} \sup_{z \in K} |f_n(z)| \le C_2.$$

Thus $|f_n(\gamma_n(t))\gamma_n'(t)| \leq C_1 \cdot C_2$ for all $n \geq n_0$, and the claim follows from Lebesgue's dominated convergence theorem. □

H 2.4 (Osgood's theorem)

Let $U \subset \mathbb{C}$ be open and $f_n : U \to \mathbb{C}$ be a sequence of holomorphic functions that converges pointwise to a function $f : U \to \mathbb{C}$. In this exercise we show that there exists an open, dense subset $U_0 \subset U$ such that the sequence f_n is locally uniformly bounded on U_0. As we will see in the course, this actually implies the local uniform convergence of f_n to f on U_0, and consequently that f is holomorphic on U_0.

(i) Let U_0 be the set of points for which $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded in a neighborhood, i.e.

$$U_0 := \{z \in U : \exists r > 0 \text{ s.t. } \sup_{n\in\mathbb{N}} \sup_{z'\in B_r(z)} |f_n(z')| < +\infty\}.$$

Show that U_0 is open.

(ii) Show that if U_0 is not dense in U, then there exists a ball $B_{r_0}(z_0) \subset U$ such that for all balls $B_{r'}(z') \subset B_{r_0}(z_0)$ the sequence f_n is not uniformly bounded on $B_{r'}(z')$.

(iii) Use (ii) to find a sequence of nested closed balls $\overline{B_{r_k}(z_k)} \subset B_{r_{k-1}}(z_{k-1})$ and a subsequence f_{n_k} such that $|f_{n_k}| \geq k$ on $\overline{B_{r_k}(z_k)}$.

(iv) Recall that due to compactness there exists an element $\bar{z}$ that belongs to all $\overline{B_{r_k}(z_k)}$. Derive a contradiction to the pointwise convergence of f_n and conclude the proof.

Solution H 2.4: (i) Let $z \in U_0$ and $r > 0$ be as in the definition of U_0. Then for all $y \in B_r(z)$ there exists $r_y > 0$ such that $B_{r_y}(y) \subset B_r(z)$. Thus

$$\sup_{n\in\mathbb{N}} \sup_{z'\in B_{r_y}(y)} |f_n(z')| \leq \sup_{n\in\mathbb{N}} \sup_{z'\in B_r(z)} |f_n(z')| < +\infty,$$

so it follows from the definition of U_0 that $y \in U_0$. Hence $B_r(z) \subset U_0$ and therefore U_0 is open.

(ii) If U_0 is not dense in U, then there exists $z_0 \in U$ and $r_0 > 0$ such that $B_{r_0}(z_0) \cap U_0 = \emptyset$. If for some ball $B_{r'}(z') \subset B_{r_0}(z_0)$ the sequence f_n were uniformly bounded on $B_{r'}(z')$, then by definition this would imply $z' \in B_{r_0}(z_0) \cap U_0$, which is a contradiction.

(iii) Keep the notation and assumptions of (ii). Since f_n is not uniformly bounded on $B_{r_0}(z_0)$, there exists $n_1 \in \mathbb{N}$ and z_1 such that $|f_{n_1}(z_1)| > 1$. Since f_{n_1} is continuous, we can find a closed ball $\overline{B_{r_1}(z_1)} \subset B_{r_0}(z_0)$ such that $|f_{n_1}(z')| > 1$ for all $z' \in \overline{B_{r_1}(z_1)}$. Using (ii) once again, the sequence f_n is not uniformly bounded on $B_{r_1}(z_1)$, so that there exists $z_2 \in B_{r_1}(z_1)$ and n_2 such that $|f_{n_2}(z_2)| > 2$. By continuity of f_{n_2} we can find a closed ball $\overline{B_{r_2}(z_2)} \subset B_{r_1}(z_1)$ such that $|f_{n_2}(z')| > 2$ for all $z' \in \overline{B_{r_2}(z_2)}$. Iterating this procedure, we find a sequence of nested closed balls $\overline{B_{r_k}(z_k)} \subset B_{r_{k-1}}(z_k)$ and a subsequence f_{n_k} such that

$$|f_{n_k}(z')| > k \quad \text{for all } z' \in \overline{B_{r_k}(z_k)}.$$

(iv) Since the sets $\overline{B_{r_k}(z_k)}$ are nested and compact, there exists an element $\bar{z}$ such that $\bar{z} \in \bigcap_{k\in\mathbb{N}} \overline{B_{r_k}(z_k)}$. For this element $\bar{z}$, we have by construction that $|f_{n_k}(\bar{z})| > k$, so that $f_n(\bar{z})$ cannot converge as $n \to +\infty$ (since clearly $n_k \to +\infty$). Hence U_0 must be dense in U. But f_n is locally uniformly bounded on U_0 by definition, which concludes the proof of the desired result. □

Remark: Note that we did not use the holomorphy of f_n in the proof. However, the latter is crucial to pass from local uniform boundedness to local uniform convergence (via Montel's theorem) in the final part of the theorem, which we did not address here.